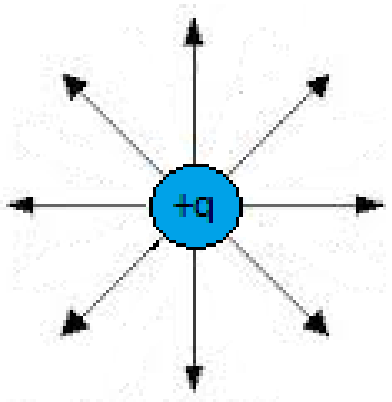
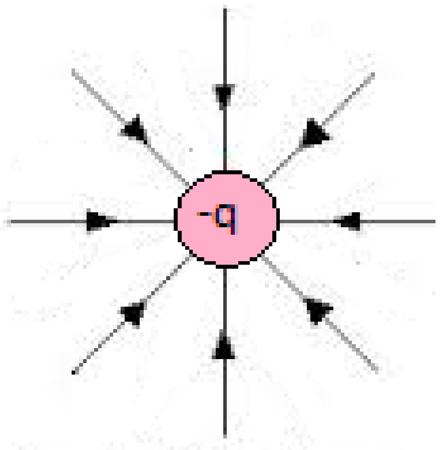
A.1 Coulomb's Law

Charges consist of two varieties: one we designate positive, and the other we designate negative, though there is nothing intrinsically 'positive' or 'negative' about them. The SI unit of charge is the Coulomb (C). For instance, the charge a proton is: e = $1.6 \times 10^{-19} C$, and the charge of an electron is: –e = -$1.6 \times 10^{-19} C$.

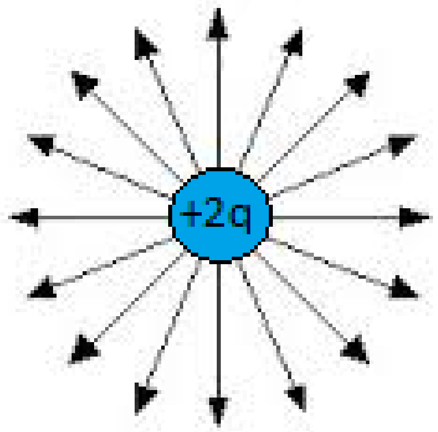
These charges exert forces on each other via an ethereal *electric field*, that every charge generates, and which pervades all space, much like how every mass also creates a gravitational field. We'd like to work out some of the features of this field. A decent physical picture is afforded by thinking of the field as consisting of 'virtual' particles that the charge is continuously radiating away from itself (if positive), or drawing into itself (if negative). It is these virtual particles that interact with neighboring charges, and via which the electric force is exerted. We can represent the trajectories of these 'virtual' particles with arrows as below. These arrows are called *electric field lines*.

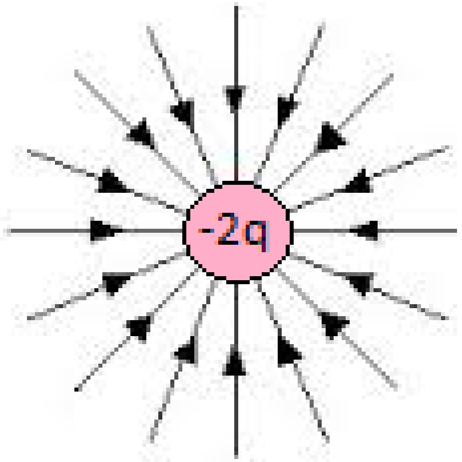




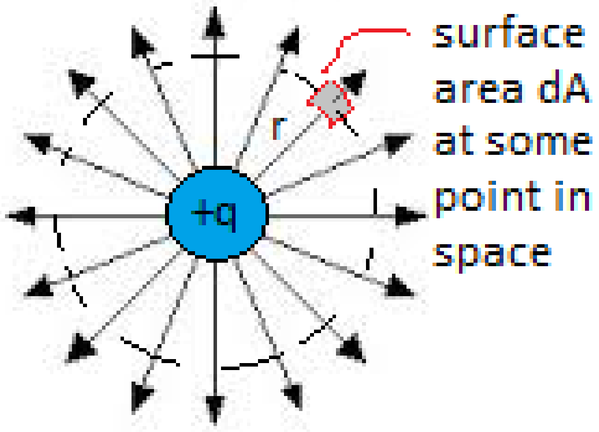
A.1 Coulomb's Law

The number of virtual particles radiated (or drawn in) is proportional to the magnitude of the charge itself. So if we double the charge, we will get double the number of virtual particles, i.e. double the number of field lines.





And the *strength* of the electric field at some point in space is proportional to the number of virtual particles (or field lines) that penetrate that region of space (in the picture I just draw one, but generally, it's much more). More specifically, it is proportional to the *density* of field lines there, i.e. the number of field lines *per m*^{2} of surface area that penetrate that region of space.



fraction of field lines that pass through dA

$$E \propto q \cdot \frac{dA / 4\pi r^2}{dA} \propto \frac{q}{4\pi r^2}$$

field lines generated is proportional to q

dividing by dA to get the density

A.1 Coulomb's Law

The constant of proportionality is $1/\varepsilon_0$, where ε_0 is called the *permittivity of free space*: $\varepsilon_0 = 8.85 \times 10^{-12} C/Nm^2$. And so we can write:

$$E = \frac{1}{\varepsilon_0}\frac{q}{4\pi r^2}$$

note this makes the units of E equal to N/C.

But then the constant $1/4\pi\varepsilon_0$ is itself defined, as k, which works out to k = $9 \times 10^9 \ Nm^2/C$. So we can write:

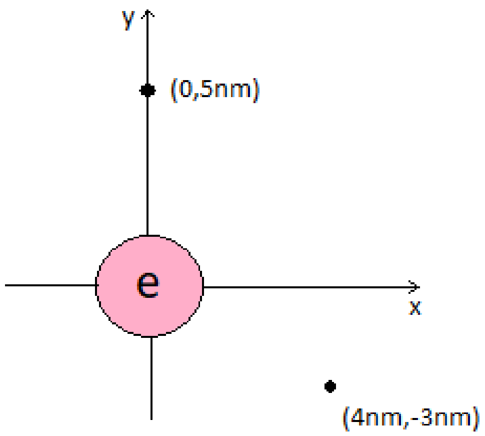
$$\mathbf{E} = \frac{k|q|}{r^2} \text{ @ directly away from positive charges, directly towards negative chares.}$$

A good website to help you visualize the electric created by an array of charges is here:

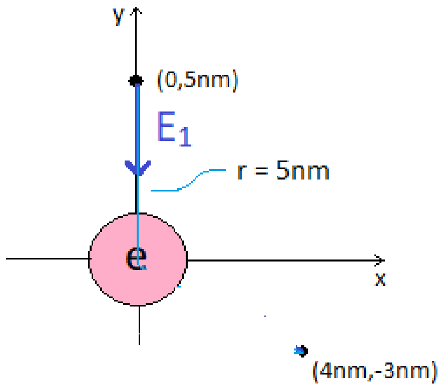
https://phet.colorado.edu/sims/html/charges-and-fields/latest/charges-and-fields_en.html

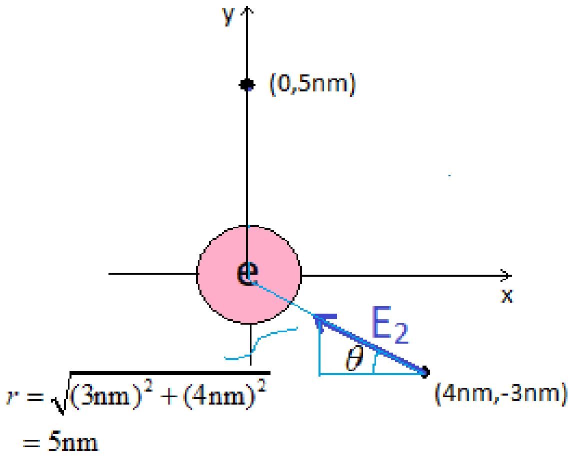
A.1 Coulomb's Law

For example, consider an electron at the origin. What is the strength and direction of the electric field at the indicated points?



First we'll draw the fields….





Then get the field strengths and directions….

$$\mathbf{E}_1 = \frac{k|q|}{r^2}(direction)$$

$$= \frac{9\times10^9(1.6\times10^{-19})}{(5\times10^{-9})^2}(-\hat{\mathbf{j}})$$

$$= -5.76\times10^7 \text{ N/C}\hat{\mathbf{j}}$$

$$\mathbf{E}_2 = \frac{k|q|}{r^2}(direction)$$

$$= \frac{(9\times10^9)\left|-1.6\times10^{-19}\right|}{(5\times10^{-9})^2}\left(-\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right)$$

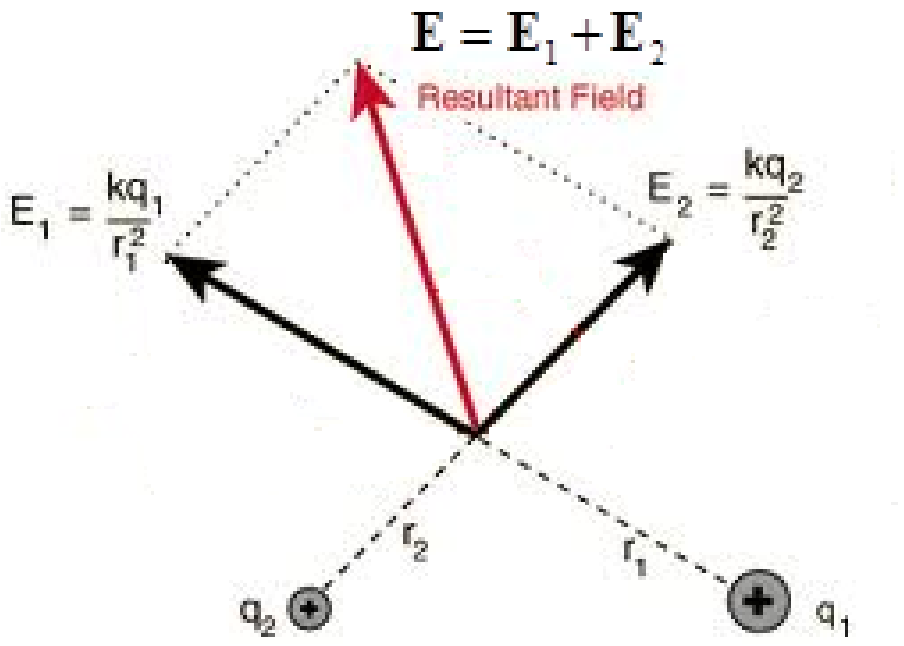
$$= \frac{(9\times10^9)\left|-1.6\times10^{-19}\right|}{(5\times10^{-9})^2}\left(-\frac{4\text{nm}}{5\text{nm}}\hat{\mathbf{i}} + \frac{3\text{nm}}{5\text{nm}}\hat{\mathbf{j}}\right)$$

$$= 5.76\times10^7 \text{ N/C}\left(-0.8\hat{\mathbf{i}} + 0.6\hat{\mathbf{j}}\right)$$

$$= -4.61\times10^7\,\hat{\mathbf{i}} + 3.46\times10^7\,\hat{\mathbf{j}} \text{ (N/C)}$$

A.1 Coulomb's Law

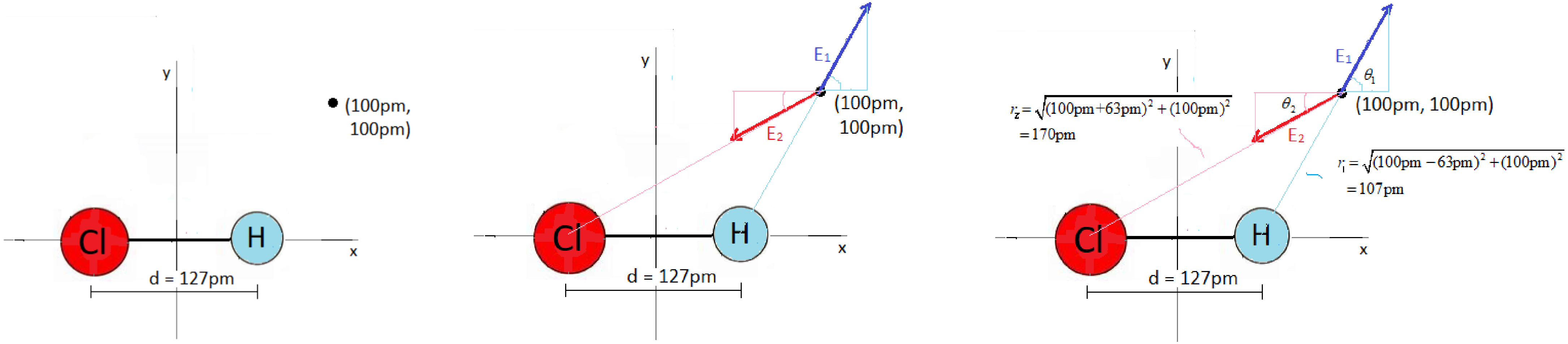
If we have multiple charges and we want the net field, we just add the fields of the respective charges.



$$\mathbf{E} = \sum_i \mathbf{E}_i \qquad \mathbf{E}_i = \frac{k|q_i|}{r_i^2}$$

A.1 Coulomb's Law

For instance, suppose we have a HCl molecule. The H atom has a charge +e, and Cl atom a charge −e. What is the field at the indicated point?



$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{k|q_1|}{r_1^2}(direction_1) + \frac{k|q_2|}{r_2^2}(direction_2) = \frac{(9\times10^9)|1.6\times10^{-19}|}{(107\times10^{-12})^2}(\cos\theta_1\hat{\mathbf{i}} + \sin\theta_1\hat{\mathbf{j}}) + \frac{(9\times10^9)|-1.6\times10^{-19}|}{(170\times10^{-12})^2}(-\cos\theta_2\hat{\mathbf{i}} - \sin\theta_2\hat{\mathbf{j}})$$

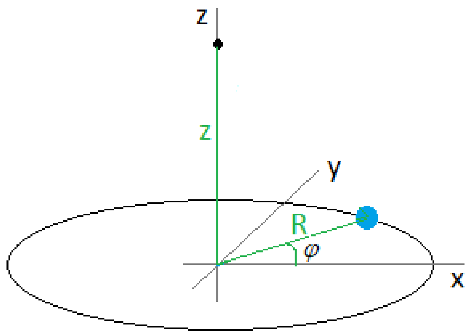
$$= 1.26\times10^{11}\left(\frac{37.5\text{pm}}{107\text{pm}}\hat{\mathbf{i}} + \frac{100\text{pm}}{107\text{pm}}\hat{\mathbf{j}}\right) + 5\times10^{10}\left(-\frac{163.5\text{pm}}{170\text{pm}}\hat{\mathbf{i}} - \frac{100\text{pm}}{170\text{pm}}\hat{\mathbf{j}}\right)$$

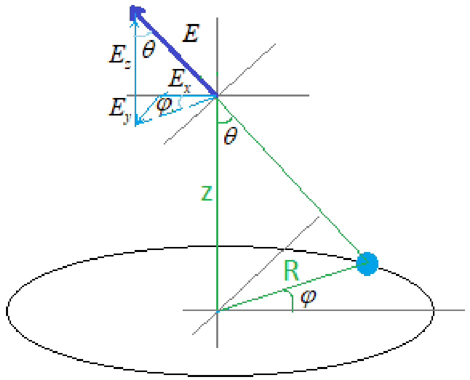
$$= (4.43\times10^{10}\,\hat{\mathbf{i}} + 1.18\times10^{11}\,\hat{\mathbf{j}}) + (-3.34\times10^{10}\,\hat{\mathbf{i}} - 2.05\times10^{10}\,\hat{\mathbf{j}}) = 1.1\times10^{10}\,\hat{\mathbf{i}} + 9.8\times10^{10}\,\hat{\mathbf{j}}$$

$$= \sqrt{(1.1\times10^{10})^2 + (9.8\times10^{10})^2}\ @\tan^{-1}\left(\frac{9.8\times10^{10}}{1.1\times10^{10}}\right)^{\circ} = 9.9\times10^{10}\ \text{N/C}\ @\,84^{\circ}\ \text{above horizontal}$$

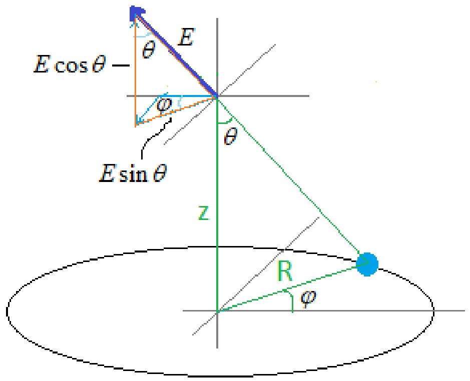
A.1 Coulomb's Law

Sometimes, we need to calculate fields in 3D. For instance consider a charge on a ring centered at the origin, and suppose we want to calculate the at some point on the z-axis. If we're given z, R, and ϕ, how do we get **E**?



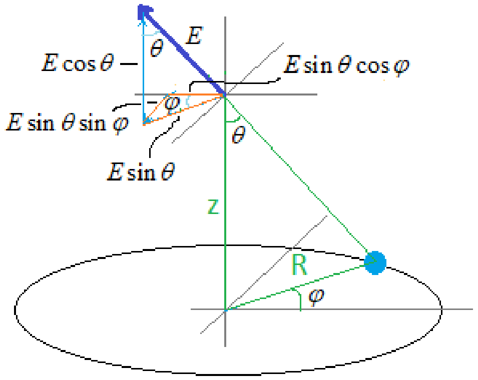


We draw the field, and its components.



First we consider the orange triangle, consisting of the field, **E**, its z-component E_z (which is the vertical line), and the projection
of E onto the x-y plane.

$$E_z = E\cos\theta$$



Then we consider the next orange triangle. If we project the EsinΘ component onto the x-y axes, then we'll get the x, y components of the field.
Negative signs are 'cause of direction.

$$E_x = -E\sin\theta\cdot\cos\varphi$$

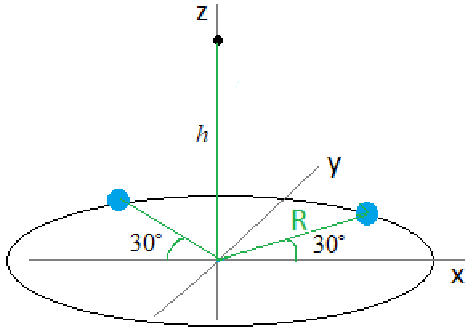
$$E_y = -E\sin\theta\cdot\sin\varphi$$

So the field is, generically:

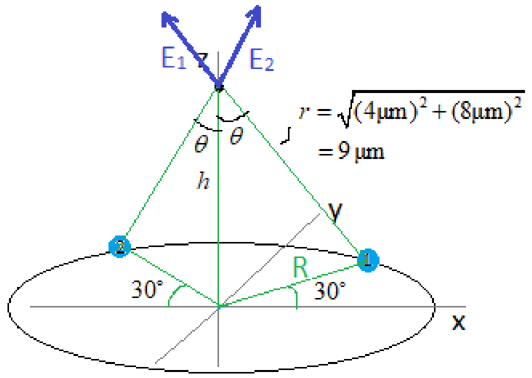
$$\mathbf{E} = E(\cos\theta\hat{\mathbf{k}} - \sin\theta\cos\varphi\hat{\mathbf{i}} - \sin\theta\sin\varphi\hat{\mathbf{j}})$$

A.1 Coulomb's Law

So for example, let's say we have arranged two 1nC charges in around a ring. Let R = 4μm, and h = 8μm. Then what is the field strength and direction at that point on the z-axis?



Well first, it's always a good idea to draw the fields. And then, we'll just use our formalism to to construct the fields….



$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = E_1(\cos\theta_1\hat{\mathbf{k}} - \sin\theta_1\cos\varphi_1\hat{\mathbf{i}} - \sin\theta_1\sin\varphi_1\hat{\mathbf{j}}) + E_2(\cos\theta_2\hat{\mathbf{k}} - \sin\theta_2\cos\varphi_2\hat{\mathbf{i}} - \sin\theta_2\sin\varphi_2\hat{\mathbf{j}})$$

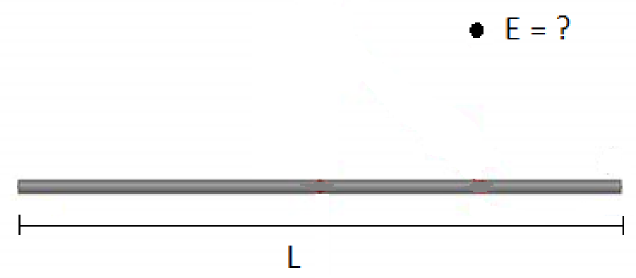
$$= \frac{k|q_1|}{r^2}\left(\frac{h}{r}\hat{\mathbf{k}} - \frac{R}{r}\cos 30^\circ\hat{\mathbf{i}} - \frac{R}{r}\sin 30^\circ\hat{\mathbf{j}}\right) + \frac{k|q_2|}{r^2}\left(\frac{h}{r}\hat{\mathbf{k}} - \frac{R}{r}\cos(150^\circ)\hat{\mathbf{i}} - \frac{R}{r}\sin(150^\circ)\hat{\mathbf{j}}\right)$$

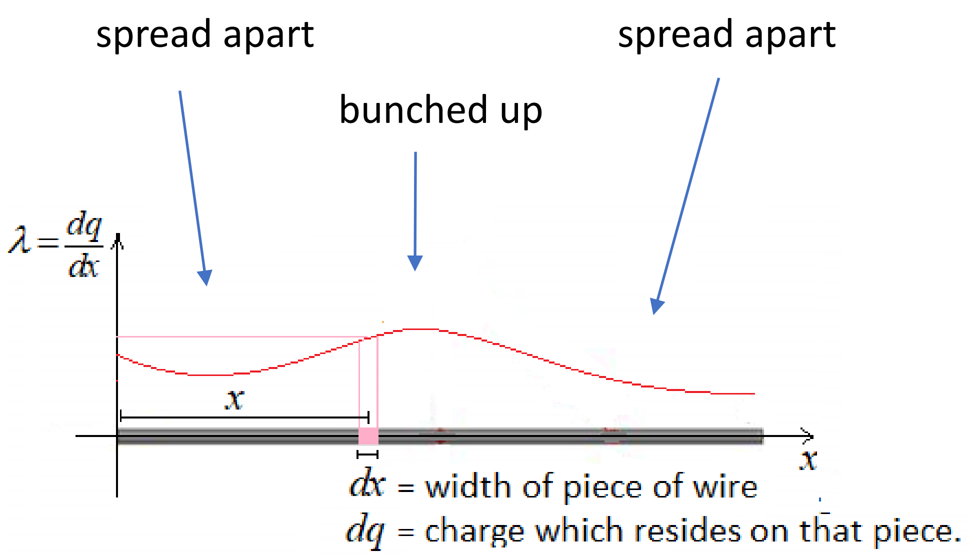
$$= \frac{(9\times10^9)(1\times10^{-9})}{(9\times10^{-6})^2}\left(\frac{8}{9}\hat{\mathbf{k}} - \frac{4}{9}\cos 30^\circ\hat{\mathbf{i}} - \frac{4}{9}\sin 30^\circ\hat{\mathbf{j}}\right) + \frac{(9\times10^9)(1\times10^{-9})}{(9\times10^{-6})^2}\left(\frac{8}{9}\hat{\mathbf{k}} - \frac{4}{9}\cos(150^\circ)\hat{\mathbf{i}} - \frac{4}{9}\sin(150^\circ)\hat{\mathbf{j}}\right)$$

$$= \frac{(9\times10^9)(1\times10^{-9})}{(9\times10^{-6})^2}\left(\frac{16}{9}\hat{\mathbf{k}} - \frac{4}{9}\hat{\mathbf{j}}\right) = 1.11\times10^{11}\ \frac{\mathrm{N}}{\mathrm{C}}\ @\ 76^\circ \text{ above -y axis}$$

A.1 Coulomb's Law

Suppose we took a rubber rod, and charged it with a cloth, like was done in lab. Then we'd have millions of charges distributed all over the rod. And say we wanted to calculate the field they all generate at some point. We would (*lit*rally) have to add up millions of electric field vectors. Not good. But we *can* do this with calculus.

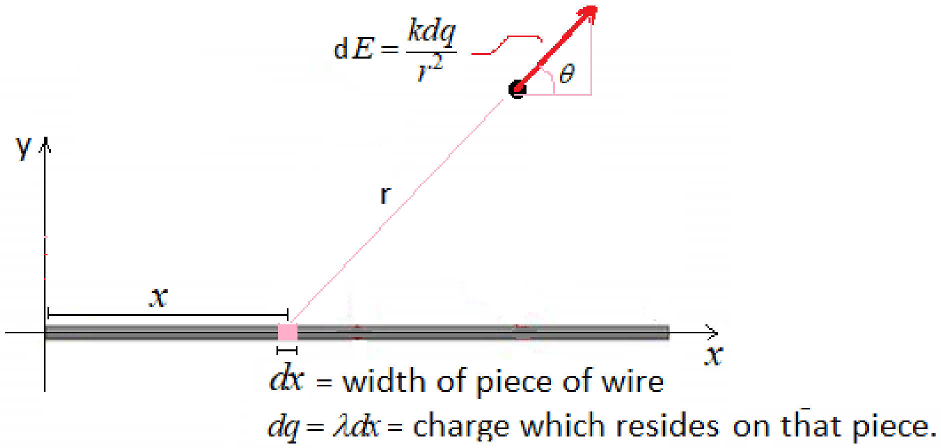




First we have to describe where the charges lie on the rod, where they're bunched up and where they're spread apart. This is done via the linear charge density function, λ(x).

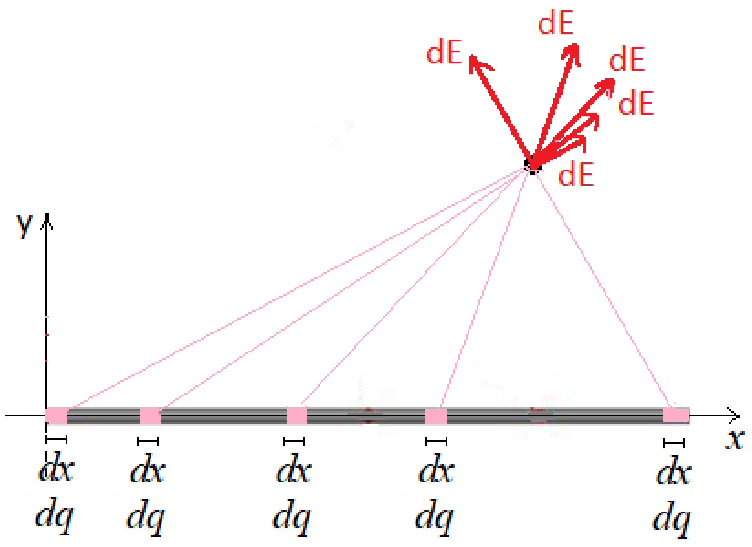
If the charge is evenly distributed along the wire, then λ would be simply q/L.

A.1 Coulomb's Law



After we've specified where the charges lie, then we calculate the field, d**E**, produced by a typical piece of charge, dq.

$$d\mathbf{E} = \frac{k|dq|}{r^2}\left[\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right]$$
$$= \frac{k|\lambda|dx}{r^2}\left[\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right]$$

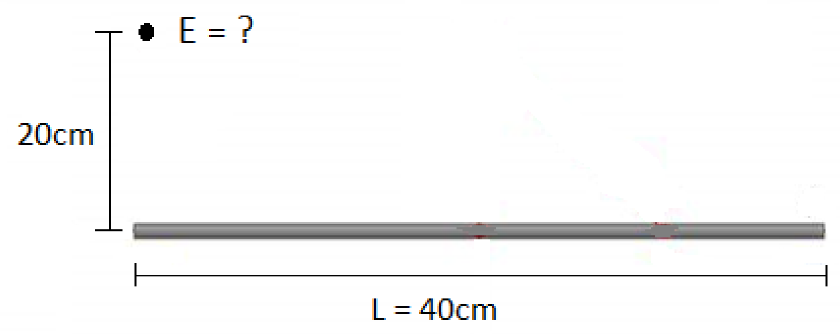


And then we simply add up (i.e. integrate) all the d**E**'s (a few such are shown) to produce the entire field **E**.

$$\mathbf{E} = \int d\mathbf{E}$$

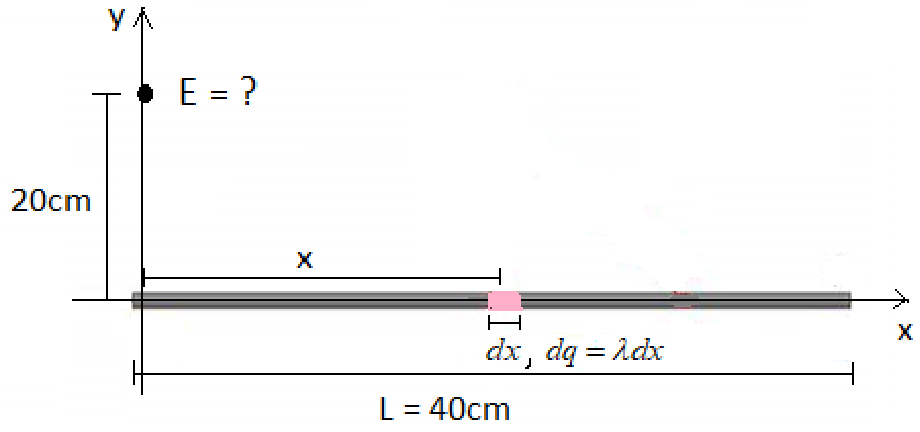
A.1 Coulomb's Law

For example, let's suppose that we deposit a charge q = -2nC uniformly on a 40cm length rod. What is the field 20cm above it's left endpoint?



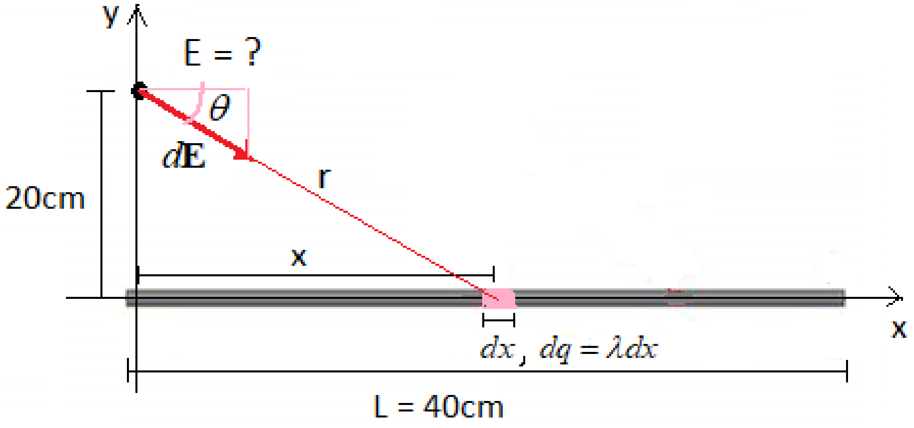
What is λ?

$$\lambda = \frac{q}{L} = \frac{-2\times10^{-9}\,\text{C}}{0.40\text{m}} = -5\times10^{-9}\,\text{C/m}$$



Isolate an arbitrary piece of the wire. Specify it's position, size, and charge.

A.1 Coulomb's Law



Calculate the field d**E** produced by the selected piece.

$$d\mathbf{E} = \frac{k|dq|}{r^2}\left[\cos\theta\hat{\mathbf{i}} - \sin\theta\hat{\mathbf{j}}\right] = \frac{k|\lambda dx|}{r^2}\left[\cos\theta\hat{\mathbf{i}} - \sin\theta\hat{\mathbf{j}}\right]$$

$$= \frac{(9\times10^9)\left|-5\times10^{-9}dx\right|}{x^2+0.20^2}\left[\cos\theta\hat{\mathbf{i}} - \sin\theta\hat{\mathbf{j}}\right]$$

$$= \frac{45dx}{x^2+0.20^2}\left[\frac{x}{\sqrt{x^2+0.20^2}}\hat{\mathbf{i}} - \frac{0.20}{\sqrt{x^2+0.20^2}}\hat{\mathbf{j}}\right]$$

$$= \frac{45x\,dx}{(x^2+0.20^2)^{3/2}}\hat{\mathbf{i}} - \frac{9dx}{(x^2+0.20^2)}\hat{\mathbf{j}}$$

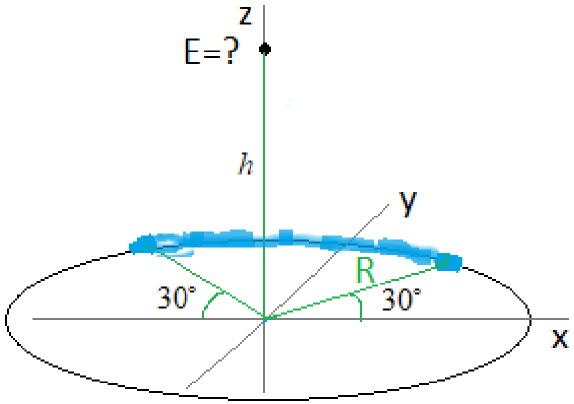
And now integrate it all to get the total field.

$$\mathbf{E} = \int d\mathbf{E} = \int_0^{L=0.40} \frac{45x\,dx}{(x^2+0.20^2)^{3/2}}\hat{\mathbf{i}} - \int_0^{L=0.40} \frac{9dx}{(x^2+0.20^2)}\hat{\mathbf{j}}$$

$$= 220\hat{\mathbf{i}} - 224\hat{\mathbf{j}} = 314\frac{\mathrm{N}}{\mathrm{C}} @\, 46^\circ \text{ below horizontal}$$

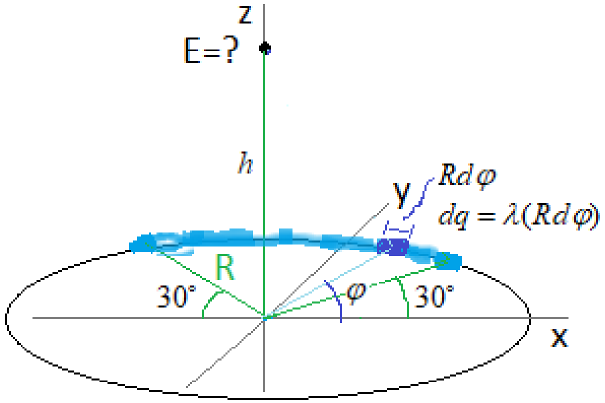
A.1 Coulomb's Law

And now let's revisit the ring (R = 4μm, h = 8μm). Suppose I uniformly smear -20nC over the length of the ring between the angles displayed. What would be the field produced by these charges, at the given point?



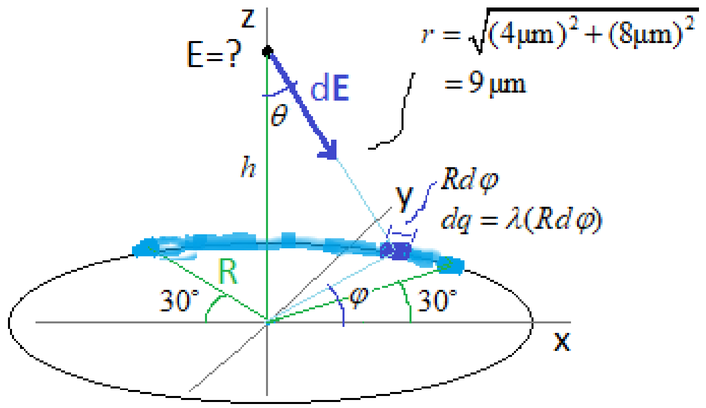
What is λ?

$$\lambda = \frac{q}{L} = \frac{-20\times10^{-9}\,\mathrm{C}}{R\Delta\theta} = \frac{-20\times10^{-9}\,\mathrm{C}}{(4\times10^{-6}\,\mathrm{m})\left(120^{\circ}\frac{\pi}{180^{\circ}}\right)} = -2.4\times10^{-3}\,\mathrm{C/m}$$



Isolate an arbitrary piece of the wire. Specify it's position, size, and charge.

A.1 Coulomb's Law



Calculate the field d**E** produced by the selected piece.

$$d\mathbf{E}=\frac{k|dq|}{r^2}(-\cos\theta\hat{\mathbf{k}}+\sin\theta\cos\varphi\hat{\mathbf{i}}+\sin\theta\sin\varphi\hat{\mathbf{j}})$$

$$=\frac{k|\lambda Rd\varphi|}{r^2}(-\cos\theta\hat{\mathbf{k}}+\sin\theta\cos\varphi\hat{\mathbf{i}}+\sin\theta\sin\varphi\hat{\mathbf{j}})$$

$$=\frac{(9\times10^{-9})\left|-2.4\times10^{-3}\cdot(4\times10^{-6})d\varphi\right|}{(9\times10^{-6})^2}\left(-\frac{8}{9}\hat{\mathbf{k}}+\frac{4}{9}\cos\varphi\hat{\mathbf{i}}+\frac{4}{9}\sin\varphi\hat{\mathbf{j}}\right)$$

$$=1.07\times10^{12}\,d\varphi\left(-\frac{8}{9}\hat{\mathbf{k}}+\frac{4}{9}\cos\varphi\hat{\mathbf{i}}+\frac{4}{9}\sin\varphi\hat{\mathbf{j}}\right)$$

And now integrate it all to get the total field.

$$\mathbf{E}=\int d\mathbf{E}$$

$$=\int_{\pi/6}^{5\pi/6}1.07\times10^{12}\left(-\frac{8}{9}\right)d\varphi\hat{\mathbf{k}}+\int_{\pi/6}^{5\pi/6}1.07\times10^{12}\left(\frac{4}{9}\right)\cos\varphi d\varphi\hat{\mathbf{i}}+\int_{\pi/6}^{5\pi/6}1.07\times10^{12}\left(\frac{4}{9}\right)\sin\varphi d\varphi\hat{\mathbf{j}}$$

$$=-2\times10^{12}\hat{\mathbf{k}}+0.82\times10^{12}\hat{\mathbf{j}}\;=2.17\times10^{12}\,\frac{\mathrm{N}}{\mathrm{C}}\,@\,22^\circ\text{ below horizontal along +y axis}$$